The Many Aspects of the Pythagorean Triangles*

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1. INTRODUCTION

The Pythagorean triangles, or triples as some call them, are right angled triangles whose sides have whole numbers a, b, c, as length, so that

$$a^2+b^2=c^2. \tag{1}$$

This equation, when generalized to exponent n instead of 2, has given us the enormous stimulus and torture of Fermat's Last Theorem.

Volumes have been written about these triangles from the number theoretic point of view and are still being written.

Equation (1) led to the study of sums of squares which are squares. This led to the definition of Pythagorean fields, in which every sum of squares is a square, and the definition of the Pythagorean number $P(F)$ of a field F, i.e. the smallest positive integer n such that every element which is a sum of squares in the field is a sum of n squares.

All of the Pythagorean triples are known to us. They are given by the expressions

$$\lambda(m^2-n^2),\quad \lambda\cdot 2mn,\quad \lambda(m^2+n^2). \tag{2}$$

Here λ, m, n are whole numbers.

The formulae (2) are a special case of a well-known identity, the 2-square identity

$$(m_1m_2-n_1n_2)^2+(m_1n_2+m_2n_1)^2=(m_1^2+n_1^2)(m_2^2+n_2^2). \tag{3}$$

*This is an extended version of the Emmy Noether lecture given by invitation of the Association of Women in Mathematics at the meeting in San Francisco, January 1981. It was supposed to be connected with part of the speaker's research.

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It is this identity which plays a vital role and will be the main theme of this article.

The identity follows easily from the multiplication rule of complex numbers:

$$(m_1+in_1)(m_2+in_2)=(m_1m_2-n_1n_2)+i(m_1n_2+m_2n_1), \tag{4}$$

and the multiplicativity of the norm of complex numbers:

$$\operatorname{norm}(\alpha\beta)=\operatorname{norm}\alpha\operatorname{norm}\beta. \tag{5}$$

But the identity (3) is also an early example of "composition of sums of squares," while Equations (4), (5) link (3) to an algebra over the reals with no divisors of zero. Hence it is no surprise that the same person who put the complex numbers into our work also started on composition of quadratic forms. That was Gauss. In the *Disquisitiones Arithmeticae*, Article 234, he says: "So far no one has considered this" (meaning composition). [In a recent book by Hlawka it is pointed out that already Vieta (1540–1603) saw a link between (3) and complex numbers.]

Both aspects of (3) will now be discussed.

2. COMPOSITION OF QUADRATIC FORMS, IN PARTICULAR THE BINARY CASE; CORRESPONDENCE OF IDEAL CLASSES AND MATRIX CLASSES; THE ABSTRACT RING CASE

Composition of two quadratic forms $b_1(x_1,\ldots,x_n)$, $b_2(x_1,\ldots,x_n)$ is carried out by multiplying these forms, obtaining a polynomial of degree 4, but then expressing this product as a quadratic form again in a new set of indeterminates which are functions of the x_i, y_k. In the classical case these functions are bilinear, but this cannot always be achieved.

The 2-square identity shows that it can be done for b_1, b_2 both sums of two squares. A famous theorem of Hurwitz shows that 2, 4, 8 are the only values of n such that two sums of n squares allow bilinear composition. Recent work by Cassels in England and Pfister in Germany extended this to $n=2^N$ if also rational composition is permitted. A matrix method introduced by the author led to a new composition rule for $n=8$ and was extended to $n=16$ by Eichhorn and Zassenhaus.

The case of forms in two variables, so called binary forms, has received particular attention. Gauss had his method in this case for integral forms. Dickson, in his *History of the Theory of Numbers III*, reports on the flood of later work.

Dedekind established a correspondence between binary quadratic forms of discriminant d and ideals in the quadratic field of discriminant d such that the composition of forms corresponds to the multiplication of ideals. The correspondence between classes of positive definite forms and narrow ideal classes is $(1,1)$.

Gauss had allowed d to be non-square-free. This leads to the case of ideals in suborders of the maximal order for the Dedekind "translation" [see also the $n=2$ case in E. C. Dade, O. Taussky, and H. Zassenhaus, *Math. Ann.* 148:31–64 (1962)].

The study of composition of binary forms over abstract rings was initiated independently by Kaplansky, Lubelski, Estes, and Butts, taken up later by Butts and Dulin, and studied for all rings by M. Kneser via quadratic modules. An article by Towber contains a study of all these achievements. There is also a paper by Estes and Earnest on composition of lattices in the same genus. Returning to the integral case, the connections with classes of matrices in the author's recent work is now mentioned.

While Dedekind linked form classes with ideal classes, the theorem of Latimer and MacDuffee links ideal classes with classes of integral matrices and hence brings the correspondence back to the rational integers.

Two $n\times n$ integral matrices A, B belong to the same matrix class if they have the same characteristic polynomial $f(x)$, which is assumed monic and irreducible (separable would be acceptable as well), and if further

$$S^{-1}AS=B, \tag{6}$$

where S is an integral unimodular matrix. A 1-1 correspondence between the classes of matrices connected with a given $f(x)$ and the ideal classes in $Q(\alpha)$ for $f(\alpha)=0$ was pointed out by Latimer and MacDuffee. It can be described by the equation

$$A\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}=\alpha\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}, \tag{7}$$

where $\alpha_1,\dots,\alpha_n$ form a Z-basis for an ideal $\mathfrak{a}$ in the order $Z[\alpha]$.

The following two ideas lead to a new composition procedure for $n=2$:

(I) Let

$$B\begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix}=\alpha\begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix}, \tag{8}$$

where $\beta_1,\ldots,\beta_n$ form a Z-basis for an ideal $\mathfrak{b}$ in $Z[\alpha]$. Assume that $\mathfrak{a}$ has an inverse $\mathfrak{a}^{-1}$. Then

$$\rho\begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix}=\mathrm{S}_{A,B}\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}, \tag{9}$$

where ρ runs through the ideal $\mathfrak{a}^{-1}\mathfrak{b}$ and $\mathrm{S}_{A,B}$ runs through *all* solutions S of (6).

(II) The set $\mathrm{S}_{A,B}$ can be expressed as a matrix whose entries are integral linear forms in two indeterminates. But this matrix can also be obtained as the product of two other matrices, one corresponding to $\mathfrak{a}^{-1}$, the other to $\mathfrak{b}$, each with entries which are integral linear forms in their respective set of two indeterminates, so that $\mathrm{S}_{A,B}$ appears as a composition of these two matrices via bilinear substitutions, derived from matrix multiplication.

Taking determinants for the three matrices mentioned, composition of quadratic forms is carried out. While the connection with ideals in (I) and (II) is used to establish the correctness of the result, the composition process can then be carried out via systems of linear integral equations.

3. ALGEBRAS OVER THE REAL NUMBER FIELD WITHOUT DIVISORS OF ZERO

As mentioned in Section 1, the special case of the complex numbers is associated with the 2-square identity (3). But even stranger and unexpected applications arise, e.g. the Cauchy-Riemann equations for two functions u_1, u_2 of two variables x_1, x_2 and the process which leads to the Laplace equation for u_1, u_2. This process is of an algebraic nature. Replacing the operators $\partial/\partial x_i$ by x_i one obtains

$$u_1x_1-u_2x_2=0, \qquad u_1x_2+u_2x_1=0, \tag{10}$$

and then one goes on to

$$u_1(x_1^2+x_2^2)=0, \qquad u_2(x_1^2+x_2^2)=0. \tag{11}$$

Interpreting these equations for u_i, x_k as real numbers, one obtains the well-known fact that the product of two complex numbers u_1+iu_2, x_1+ix_2 cannot be zero unless one of the factors is zero.

However, the author applied the above argument of hers to generalized Cauchy-Riemann equations for n functions of m variables, assuming that they satisfy n linear partial differential equations. Assume further that these equations are of such a nature that the Laplace equations emerge for all of them by suitable combinations when multiplied by linear forms in the operators. It was then observed that this would imply the existence of an algebra over the reals without divisors of zero. In the meantime it was shown (by Bott, Kervaire, Milnor, Adams) that this implies that n is one of the numbers $1,2,4,8$. For $n=8$ such an algebra has to be nonassociative by a theorem of Frobenius. This theorem too can be connected with the 2-square identity. For (3) implies that the points of the unit sphere will form a group under multiplication. E. Cartan had shown that the points of the n-sphere $x_1^2 + \cdots + x_n^2 = 1$ are a group space only for $n = 1,2,4$. Linking the n-dimensional sphere to algebras over the reals with n basis elements, the author re-proved Frobenius's theorem.[1]

The result concerning generalized Cauchy-Riemann equations was reproved by Stiefel. It also leads to a statement about matrix equations which was later rediscovered by combinatorialists.

4. PYTHAGORAS' THEOREM VERSUS THE PYTHAGOREAN TRIANGLES

The Pythagorean triangles are a very early example of a problem that comes up very frequently. Given a fact that is valid over fields, will it also be valid over rings, and if it is not valid over a specific ring, for what cases will it be, or alternatively, for what rings will it be valid? A few examples are as follows:

EXAMPLE 1. A quadratic form over the reals can be transformed to $\Sigma a_i x_i^2$ by a substitution of its variables. When is this true for a quadratic form over the ring of integers?

A positive definite real form in n indeterminates can be transformed into a sum of n squares of linear forms. But even a binary integral quadratic form may need as many as five sums of squares of integral linear forms. Mordell showed that for sufficiently large n an integral positive definite quadratic form in n indeterminates may not be expressible at all as a sum of integral squares.

[1] Frobenius's proof uses matrices which he derives from bilinear forms and their determinants. The term matrix was introduced by Cayley in 1857. But Frobenius speaks about "systems of n^2 quantities, arranged into n rows and n columns." Other early re-proofs of Frobenius's theorem were given by E. Cartan and by Dickson.

EXAMPLE 2. It is known that a matrix with entries from a field can be expressed as the product of 2 symmetric matrices with elements from the same field. The author then asked the question: what happens if the entries of the matrix are rational integers? Can the factors then be chosen integral too? She showed that the matrix

$$\begin{pmatrix} -8 & 3 \\ 5 & 8 \end{pmatrix}$$

cannot be factored in this way and characterized the 2×2 matrices for which it is possible. There is a connection there with ideal classes in quadratic fields.

EXAMPLE 3. In recent years much attention is given to the level (or *Stufe*) of not formally real fields, i.e. the smallest integer n required to express -1 as a sum of n squares in the field. In recent years it was shown that n is a power of 2. Now there is quite a group of people working on the levels of rings.

5. THE FACTORIZATION OF m^2+n^2 AND OF DETERMINANTS OF INTEGRAL MATRICES

For integral values of m, n one can study the factorization of the sum of two squares, not a square itself, into sums of two integral squares, instead of the composition—i.e., the question of when

$$m^2+n^2=\left(m_1^2+n_1^2\right)\left(m_2^2+n_2^2\right), \qquad m_i, n_i \neq 0. \tag{12}$$

The latter condition is automatically fulfilled if m^2+n^2 has a square factor coming from a Pythagorean triangle. Using integers in $Q(i)$, the so-called Gaussian integers, the above factorization is equivalent with the following statement: Let $m^2+n^2=a\,b$, $a, b \in Z$ and both sums of two squares. Then there exist Gaussian numbers α, β such that norm $\alpha=a$, norm $\beta=b$, and $\alpha\beta=m+in$.

An analogous statement is correct for sums of 4 squares using integral quaternions: either the Lipschitz quaternions, which have integral coefficients, or the Hurwitz quaternions. No condition is then required on the factors a, b.

An analogous statement is further correct for a large class of cases of 8 squares, connected with the various definitions of integral Cayley numbers (for these definitions see Estes and Pall). The case of Cayley numbers with integral coefficients was studied by Benneton, and later again by Pall and the author with a different method; the other rings of integer Cayley numbers were treated by Feaux and Hardy.

There are other examples where the factorization of "norms" implies the factorization of the elements coming from a system with multiplication rule and multiplicative "'norm." This applies particularly to integral matrices and their determinants. For matrices with entries in a field it is a trivial observation that $\det A = rs \neq 0$ implies $A = B \cdot C$ where $\det B = r$, $\det C = s$. The same can be shown to be true for matrices with entries in Z, via the Smith normal form. However, the same is not true for arbitrary rings. Recently Estes and Matijevic characterized a class of rings for which it holds.

Another example comes from integral circulants, or more generally integral group matrices, and can also be formulated as a problem concerning the integral group ring. The following question was posed by the author: When can an integral circulant C with $\det C = rs \neq 0$ be expressed as a product of integral circulants C_1, C_2 with determinants r, s? This question is even harder, for there is not always an integral $n \times n$ circulant C with a given $\det C$. The problem was studied by M. Newman, by Newman and Mahoney, in the thesis of Mahoney, and by Tai, Zong Duo and Feng, Xu Ning, (unpublished).

6. SOME SPECIAL RESULTS CONCERNING PYTHAGOREAN TRIANGLES

It is futile to attempt a survey of the triangles. This report aims to demonstrate that large areas of research could have been motivated by them, but there is no claim that they actually were.

From the more recent work concerning the triangles themselves a small sample is now given:

(1) *A modern proof for the formulae* (2) is given by Taussky [62]. This could have been obtained by Emmy Noether and was quite possibly known to her. It uses Galois cohomology, namely Hilbert's Theorem 90.

(2) *The maximum real subfield of the cyclotomic field* $Q(\zeta_p)$, ζ_p *a p-th root of* 1, p *a prime.* The field $Q(\zeta_p)$ is an extension of degree 2 of its maximum real subfield. The author had found a condition which ensures that a p-dimensional integral unimodular positive definite circulant C can be expressed as C_1C_1' with C_1 again an integral circulant. The condition is that the characteristic roots of C are norms from $Q(\zeta_p)$ to $Q(\zeta_p + \bar{\zeta}_p)$. The case

when these norms are merely squares in the real field is of special significance. This could be a Pythagorean triple situation. A thesis by D. Davis is related to this problem. Norms from quadratic fields other than $Q(i)$ can also be squares, e.g. $7^2-10\times 2^2=3^2$.

(3) *An application to imaginary quadratic number fields whose class number is divisible by* 16. A famed result of Gauss concerns the connection between the number of representations of an integer m as a sum of three squares and the ideal class number of $Q(\sqrt{-m})$ (expressed by Gauss in the language of quadratic forms). More recently this fact was re-proved by Venkov using the tools of Hurwitz quaternions (see also a more modern version by H. P. Rehm). This technique was employed by Hanlon and Morton to the case of a prime which is a sum of two squares one of which comes from a Pythagorean triangle.

(4) *The equation $a^2+b^2=mc^2$, m square free* The Pythagorean triangles are a special case of this. It leads to the norm equations

$$a^2-mc^2=-b^2$$

and the representation of -1 as a norm in $Q(\sqrt{m})$.

(5) *A 3×3 matrix which transforms arbitrary Pythagorean triples into other Pythagorean triples.*

(6) *A link between Pythagorean triangles and celestial mechanics.* Changing the subject completely, a link between the formulae (2) and celestial mechanics, proved by Levi-Civita and used by Stiefel and Scheifele, Springer 1971, was reported to the author by Stiefel: The expressions

$$x_1=u_1^2-u_2^2, \qquad x_2=2u_1u_2, \qquad r=u_1^2+u_2^2 \tag{13}$$

define a mapping from the u_1, u_2 plane to the x_1, x_2 plane with $r=\sqrt{x_1^2+x_2^2}$ the distance from 0 to (x_1, x_2). The following can then be shown: the line l given by $u_1=c$, c a constant, is mapped on to a parabola, and if (u_1, u_2) moves, in a certain time scale, with constant speed along l, then (x_1, x_2) moves in a parabolic path according to the Kepler laws.

(7) Finally, studying the mapping (13), Hlawka made an interesting application, obtaining an approximation of right angled triangles by the Pythagorean triangles.

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